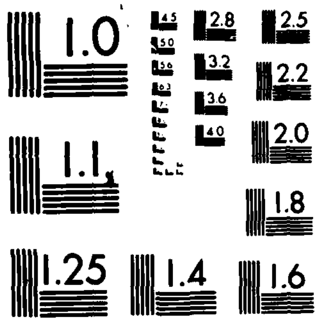
AD-A139 312   ON THE ALMOST PERIODICITY OF THE SOLUTIONS OF AN INTEGRODIFFERENTIAL EQUATION(U) WISCONSIN UNIV-MADISON MATHEMATICS RESEARCH CENTER   O J STAFFANS FEB 84   1/1

UNCLASSIFIED   MRC-TSR-2642 DAAG29-80-C-0041   F/G 12/1   NL

END
DATE
FILMED
4-84
DTIC


1.0
2.8
2.5
3.2
2.2
3.6
4.0
2.0
1.1
1.8
1.25
1.4
1.6

MRC Technical Summary Report #2642

ON THE ALMOST PERIODICITY OF THE SOLUTIONS OF AN INTEGRODIFFERENTIAL EQUATION

Olof J. Staffans

Mathematics Research Center
University of Wisconsin—Madison
610 Walnut Street
Madison, Wisconsin 53705

February 1984

(Received January 19, 1984)

Sponsored by

U. S. Army Research Office
P. O. Box 12211
Research Triangle Park
North Carolina 27709

UNIVERSITY OF WISCONSIN - MADISON
MATHEMATICS RESEARCH CENTER

ON THE ALMOST PERIODICITY OF THE SOLUTIONS OF AN INTEGRODIFFERENTIAL EQUATION

Olof J. Staffans

Technical Summary Report #2642

February 1984

ABSTRACT

This paper ~~We~~ discusses the almost periodicity of bounded solutions of the integrodifferential equation

$$x' + \mu * x = f.$$

Here x and f map $\mathbb{R}$ into $\mathbb{C}^n$, the prime denotes differentiation, μ is an n by n matrix valued finite measure on $\mathbb{R}$, and f is either an almost periodic distribution, or an almost periodic function in the sense of Bohr, Stephanoff, Weyl or Besicovitch. In the first three cases ~~we~~ the author gives a simple sufficient condition (countability of the set where the characteristic function of the kernel is not invertible) for bounded solutions to be almost periodic. This condition is no longer sufficient in the last two cases, as ~~we~~ is shown with a simple counterexample.

AMS (MOS) Subject Classifications: 45J05, 45A05, 45E10, 42A75

Key Words: Almost periodic, convolution equation, integro-differential equation

Work Unit Number 1 - Applied Analysis

Sponsored by the United States Army under Contract No. DAAG29-80-C-0041.

SIGNIFICANCE AND EXPLANATION

This paper discusses linear systems of equations of the form

$$(*) \qquad \frac{d}{dt}x(t) + \int_{-\infty}^{\infty}[d\mu(s)]x(t-s) = f(t), \; -\infty < t < \infty \; .$$

Typical examples of such equations are ordinary differential equations, differential delay equations, retarded functional differential equations, and integrodifferential equations. Our main objective consists of finding conditions which imply that all bounded solutions are almost periodic, i.e. they can be approximated by sums of periodic functions. It is shown that in most cases bounded solutions of (*) are almost periodic, provided one uses a sufficiently strong notion of almost periodicity.

The responsibility for the wording and views expressed in this descriptive summary lies with MRC, and not with the author of this report.

ON THE ALMOST PERIODICITY OF THE SOLUTIONS OF AN INTEGRODIFFERENTIAL EQUATION

Olof J. Staffans

1. Introduction

In this paper we study the integrodifferential equation

$$(1.1) \qquad x' + \mu * x = f.$$

Here x and f are defined on $\mathbf{R}$ and take column vector values in $\mathbf{C}^n$, μ is an n by n matrix valued finite measure on $\mathbf{R}$, and the equation is supposed to hold in the distribution sense (the prime stands for differentiation). More specifically, we ask the following question: If f is almost periodic in some sense, then is it in general true that every bounded distribution solution x of (1.1) is also almost periodic in the same sense, or possibly in some other sense?

The main purpose of this paper is to present sufficient conditions which imply that bounded solutions are almost periodic. We essentially restrict ourselves to four notions of almost periodicity, namely almost periodicity in the distribution sense, in the Bohr and Stephanoff sense, and almost periodicity with an absolutely convergent Fourier series. In particular, we exclude the Weyl and Besicovitch classes of almost periodic functions. There is a very good reason for doing so; we show with a counterexample that our main result cannot be extended to these two function classes.

Sponsored by the United States Army under Contract No. DAAG29-80-C-0041.

The main theorems are presented in Section 2. First a necessary condition is given for (1.1) to have almost periodic solutions, and then sufficient conditions are discussed which imply that all bounded solutions of (1.1) are almost periodic in one sense or another. The counterexample which we mentioned above is also stated here.

Section 3 contains a discussion on almost periodic distributions and Bohr almost periodic functions as well as a number of preliminary lemmas. Proofs of theorems related to distribution and Bohr almost periodicity are given in Section 4. Section 5 starts with a short discussion of Stephanoff almost periodicity, and continues with proof of a theorem which applies when f in (1.1) is Stephanoff almost periodic. Finally, in Section 6 we define Weyl and Besicovitch almost periodicity, and show that our main result cannot be extended to these two function spaces.

Acknowledgement

The author wants to thank Professor C. Corduneanu for drawing our attention to the problem discussed here at the AMS meeting in Evanston in November 1983. The main results af this paper were obtained during a subsequent visit to the Mathematics Research Center at the University of Wisconsin, Madison.

2. Statement of results

The question which we want to discuss is the following: When is it true that all bounded solutions of (1.1) are almost periodic?

A natural way of beginning the discussion is to first check what the necessary conditions are which have to be satisfied for (1.1) to have an almost periodic soluton. One straightforward result in this direction is the following:

Theorem 1. *Let* x *be an almost periodic* C^n*-valued distribution, let* μ *be a finite* n *by* n *matrix valued measue on* $\mathbf{R}$, *and define* f *by* (1.1). *Then* f *is an almost periodic distribution. Moreover, the Fourier series* $\sum_{k=1}^{\infty} b_k e^{i\omega_k t}$ *of* f *satisfies*

$$\sum_{k=1}^{\infty} b_k e^{i\omega_k t} = \sum_{k=1}^{\infty} D(\omega_k) a_k e^{i\omega_k t}, \tag{2.1}$$

where $\sum_{k=1}^{\infty} a_k e^{i\omega_k t}$ *is the Fourier series of* x,

$$D(\omega) = i\omega I + \hat{\mu}(\omega), \quad \omega \in \mathbf{R}, \tag{2.2}$$

I *is the identity matrix, and* $\hat{\mu}(\omega) = \int_{\mathbf{R}} e^{i\omega t} d\mu(t)$.

Clearly, Theorem 1.1 gives us a necessary condition on f for the equation (1.1) to have an almost periodic distribution solution. First of all, f has to be an almost periodic distribution. Moreover, the Fourier series $\sum_{k=1}^{\infty} b_k e^{i\omega_k t}$ of f cannot be completely arbitrary, but it has to be of the form (2.1). In particular, if $D(\omega_k)$ is not invertible for some ω_k, then the requirement that b_k should belong to the range of $D(\omega_k)$

restricts b_k to a proper subspace of C^n. For example, in the scalar case b_k has to vanish whenever $D(\omega_k)$ vanishes.

Suppose that f is an almost periodic distribution, and that the coefficients b_k in its Fourier series $\sum_{k=1}^{\infty} b_k e^{i\omega_k t}$ belong to the range of $D(\omega_k)$ for all k. In general it is still true that there may exist unbounded distribution solutions which satisfy (1.1) in some sense, and these cannot possibly be almost periodic, because every almost periodic distribution is bounded in the distribution sense. However, let us suppose that x is bounded. Is it then automatically true that x is almost periodic? The answer to this question dependes on the size of the set Z defined by

$$(2.3) \qquad Z = \{\, \omega \in \mathbf{R} \mid D(\omega) \text{ is not invertible} \,\}.$$

If Z is uncountable, then, at least in the scalar case, one can always construct a smooth bounded solution x to the equation $x' + \mu * x = 0$ which is not almost periodic (see [8, pp. 300 - 301]). This implies that although (1.1) may have almost periodic solutions, it also has solutions which are not almost periodic (or it has no solutions whatsoever). On the other hand, if Z is countable, then all bounded solutions are almost periodic:

Theorem 2. *Let μ be a finite n by n matrix valued measure, Define D and Z by (2.2) and (2.3), and suppose that Z is countable. Let f be an almost periodic distribution. Then every bounded distribution solution x of (1.1) is an almost periodic distribution.*

Observe that no claim is made about the existence of a bounded solution.

Our proof of Theorem 2 is based on a reduction to the following result, which has been essentially known for approximately twenty years (cf. [5] and [9]): If the set Z is countable, and if f is almost periodic in the sense of Bohr, then every bounded solution x of (1.1) is also almost periodic in the same sense. As a matter of fact, this result is true even in a slightly stronger form:

Theorem 3. Let μ be of the same type as in Theorem 2, with Z countable, and let f be almost periodic in the sense of Stephanoff. Then every bounded *distribution* solution x of (1.1) is almost periodic in the sense of Bohr.

In other words, the solution x of (1.1) has better smoothness properties than the input function f.

Another class of almost periodic functions is the class of functions x of the form $x(t) = \sum_{k=1}^{\infty} a_k e^{i\omega_k t}$, $t \in \mathbf{R}$, where $\sum_{k=1}^{\infty} |a_k| < \infty$. We call functions x of this type almost periodic with an absolutely convergent Fourier series, or simply "absolutely convergent almost periodic". Clearly, by Theorem 1, if one wants a solution x of (1.1) to be an absolutely convergent almost periodic function, then necessarily the Fourier series $\sum_{k=1}^{\infty} b_k e^{i\omega_k t}$ of f must satisfy

$$(2.4) \qquad \sum_{\substack{k=1 \\ \omega_k \notin Z}}^{\infty} |D^{-1}(\omega_k) b_k| < \infty.$$

One can show with simple counterexamples that if Z is infinite, then this condition is not in general sufficient for all bounded solutions of (1.1) to be absolutely convergent almost periodic. However, if Z is finite, then it is so:

Theorem 4. *Let* μ *be a finite* n *by* n *matrix valued measure on* $\mathbf{R}$, *define* D *and* Z *by* (2.2) *and* (2.3), *and suppose that* Z *is finite. Let* f *be an almost periodic distribution with a Fourier series* $\sum_{k=1}^{\infty} b_k e^{i\omega_k t}$ *satisfying* (2.4). *Then every bounded distribution solution* x *of* (1.1) *is an absolutely convergent almost periodic function*.

So far we have throughout assumed that f in (1.1) is an almost periodic distribution. There exist Weyl and Besicovitch almost periodic functions which are not almost periodic in the distribution sense, and the question arises whether Theorem 2 is also true for these two classes of almost periodic functions. The answer is "no", as the following counterexample shows (to improve the readability we use a rather imprecise formulation; a more exact formulation is given in Section 5):

Theorem 5. *Let* f *be a continuous function with compact support, and suppose that* $\int_{\mathbf{R}} f(t)dt \neq 0$. *Then* f *is Weyl almost periodic and its integral* $\int_{-\infty}^{t} f(s)ds$ *is a bounded uniformly continuous function, which is not almost periodic even in the sense of Besicovitch*.

Every Weyl almost periodic function is Besicovitch almost periodic, and therefore Theorem 5 provides a counterexample to Theorem 2 with the class of almost periodic distributions replaced by the classes of Weyl and Besicovitch almost periodic functions (taking $\mu \equiv 0$ we get $D(\omega) = i\omega I$; this function is invertible for every $\omega \neq 0$).

3. Almost Periodic Distributions and Bohr Almost Periodic Functions

To shorten the presentation, let us introduce some notations. We let $\mathbf{B}(\mathbf{R};\mathbf{C}^n)$ denote the space of Schwartz bounded $\mathbf{C}^n$-valued distributions on $\mathbf{R}$, and $\mathbf{AP}(\mathbf{R};\mathbf{C}^n)$ the space of Schwartz almost periodic distributions. A reader unfamiliar with these two concepts may consult e.g. Schwartz's book [10] (which uses a different notation). The space of bounded, uniformly continuous $\mathbf{C}^n$-valued functions on $\mathbf{R}$ is denoted $BUC(\mathbf{R};\mathbf{C}^n)$, and $BAP(\mathbf{R};\mathbf{C}^n)$ stands for the set of Bohr almost periodic functions.

We denote the set of finite n by n matrix valued measures on $\mathbf{R}$ by $M(\mathbf{R};\mathbf{C}^{n\times n})$. If $\mu \in M(\mathbf{R};\mathbf{C}^{n\times n})$ and either $f \in \mathbf{B}(\mathbf{R};\mathbf{C}^n)$ or $f \in BUC(\mathbf{R};\mathbf{C}^n)$, then the convolution $\mu * f$ is well defined, and it belongs to the same space as f (see [10, p. 203] for the distribution case).

Our proofs of statements concerning distributions in $\mathbf{B}(\mathbf{R};\mathbf{C}^n)$ and $\mathbf{AP}(\mathbf{R};\mathbf{C}^n)$ are based on a reduction to the corresponding statements concerning functions which belong to $BUC(\mathbf{R};\mathbf{C}^n)$ and to $BAP(\mathbf{R};\mathbf{C}^n)$, respectively. One gets from the former case to the latter by convolving the distributions with a sufficiently smooth function. Define

$$(3.1) \qquad e(t) = \begin{cases} e^{-t}, & t > 0, \\ 0, & t < 0. \end{cases}$$

Then e is a L^1-function whose Fourier transform

$$(3.2) \qquad \hat{e}(\omega) = (1 + i\omega)^{-1}, \quad \omega \in \mathbf{R}.$$

vanishes nowhere. We denote the m-fold convolution of e with itself by e^{m*}, $m \geq 1$. Explicitly,

$$e^{m*}(t) = \begin{cases} \dfrac{t^{m-1}}{(m-1)!} e^{-t}, & t \geq 0, \\ 0, & t < 0, \end{cases}$$

and the Fourier transform $(e^{m*})^\wedge$ of e^{m*} is $(e^{m*})^\wedge(\omega) = [\hat{e}(\omega)]^m = (1 + i\omega)^{-m}$, $\omega \in \mathbf{R}$. We define e^{0*} to be equal to δ, the unit point mass at zero. For each $m \geq 0$, the convolution operator which takes x into $e^{m*} * x$ maps $\mathbf{B}(\mathbf{R};\mathbf{C}^n)$ bicontinuously onto itself. Its inverse is the operator $(1 + \frac{d}{dt})^m$.

<u>Lemma 3.1</u>. <u>Let</u> $x \in \mathbf{B}(\mathbf{R};\mathbf{C}^n)$. <u>Then there exists an integer</u> $m \geq 0$ <u>such that</u> $e^{m*} * x \in \mathbf{BUC}(\mathbf{R};\mathbf{C}^n)$. <u>Moreover</u>, $x \in \mathbf{AP}(\mathbf{R};\mathbf{C}^n)$ <u>if and only if</u> $e^{m*} * x \in \mathbf{BAP}(\mathbf{R};\mathbf{C}^n)$ <u>for some integer</u> $m \geq 0$.

This lemma can be deduced e.g. from [10, (VI,8;6), pp. 205 and 207] (Schwartz uses the kernel $e^{-|t|}$, $t \in \mathbf{R}$, instead of our kernel e, but the argument remains the same).

One can define the Fourier series of a function $x \in \mathbf{AP}(\mathbf{R};\mathbf{C}^n)$ in the following way. By Lemma 3.1, $e^{m*} * x \in \mathbf{BAP}(\mathbf{R};\mathbf{C}^n)$, and it has a Fourier series $\sum_{k=1}^{\infty} b_k e^{i\omega_k t}$ in the sense of Bohr. We define the Fourier series of a to be $\sum_{k=1}^{\infty} (1 + i\omega_k)^m b_k e^{i\omega_k t}$. That this definition is independent of the particular value of m which was chosen follows from the following well known result, and from (3.2):

Lemma 3.2. *Let* $\mu \in M(R;C^{n\times n})$, *and let* $x \in BAP(R;C^n)$ *have the Fourier series* $\sum_{k=1}^{\infty} a_k e^{i\omega_k t}$. *Then* $\mu * x \in BAP(R;C^n)$, *and the Fourier series of* $\mu * x$ *is* $\sum_{k=1}^{\infty} \hat{\mu}(\omega_k) a_k e^{i\omega_k t}$, *where* $\hat{\mu}$ *is the Fourier transform of* μ.

Let us introduce one more notation. When we write $x \simeq \sum_{k=1}^{\infty} a_k e^{i\omega_k t} \in AP(R;C^n)$, we mean that x is an almost periodic distribution with Fourier series $\sum_{k=1}^{\infty} a_k e^{i\omega_k t}$. With this notation, the following corollary to Lemmas 3.1 and 3.2 can be written in a very compact form:

Lemma 3.3. *Let* $\mu \in M(R;C^{n\times n})$, *and let* $x \simeq \sum_{k=1}^{\infty} a_k e^{i\omega_k t} \in AP(R;C^n)$. *Then* $\mu * x \in AP(R;C^n)$, *and* $\mu * x \simeq \sum_{k=1}^{\infty} \hat{\mu}(\omega_k) a_k e^{i\omega_k t}$, *where* $\hat{\mu}$ *is the Fourier transform of* μ.

This is true, because $e^{m*} * \mu * x = \mu * e^{m*} * x$, and Lemmas 3.1 and 3.2 can be applied.

In our proofs of Theorems 1 - 4 it is convenient to rewrite (1.1) in a form which does not involve a differentiation:

Lemma 3.4. *Let* $\mu \in M(R;C^{n\times n})$. *Then* $x \in B(R;C^n)$ *is a solution of* (1.1) *if and only if it satisfies*

$$(3.3) \qquad x + a * x = e * f,$$

where e *is the function defined in* (3.1), *and*

$$(3.4) \qquad a = e * \mu - eI.$$

Proof. If $x \in B(R;C^n)$ satisfies (1.1), then (1.1) implies $f \in B(R;C^n)$. Convolving (1.1) with e in the distribution sense we get

$$e * x' + e * \mu * x = e * f,$$

or equivalently

$$e' * x + e * \mu * x = e * f.$$

As $e' = \delta - e$, we get (3.3), (3.4).

Conversely, let $x \in B(R;C^n)$ satisfy (3.4). Differentiate (3.4) to get

$$x' + a' * x = e' * f = f - e * f.$$

Now

$$a' = e' * \mu - e'I = \mu - e * m - \delta I + eI = \mu - \delta I - a,$$

so we get

$$x' + \mu * x - x - a * x = f - e * f.$$

Adding (3.3) to this equation we get (1.1). ∎

4. Proofs of Theorems 1, 2 and 4

After the preliminary considerations in Section 3 the proof of Theorem 1 is very easy:

Proof of Theorem 1. Let $x \simeq \sum_{k=1}^{\infty} a_k e^{i\omega_k t} \in \mathbf{AP}(\mathbf{R};\mathbf{C}^n)$ be a solution of (1.1). Then, by Lemma 3.4, it satisfies (3.3). By (3.3) and Lemma 3.3, $e * f \in \mathbf{AP}(\mathbf{R};\mathbf{C}^n)$, and

$$e * f \simeq \sum_{k=1}^{\infty} (\delta I + a)\hat{\ }(\omega_k) a_k e^{i\omega_k t}.$$

We know that $f \in \mathbf{B}(\mathbf{R};\mathbf{C}^n)$ (see [10, Theorem XXVI, p. 203]). It follows from Lemma 3.1 that $f \in \mathbf{AP}(\mathbf{R};\mathbf{C}^n)$, and by Lemma 3.2

$$f \simeq \sum_{k=1}^{\infty} [\hat{e}(\omega_k)]^{-1}(\delta I + a)\hat{\ }(\omega_k) a_k e^{i\omega_k t}.$$

Now $\hat{e}(\omega) = (1 + i\omega)^{-1}$, $\omega \in \mathbf{R}$, and

$$(4.1) \qquad (\delta I + a)\hat{\ }(\omega) = I + \hat{e}(\omega)(\hat{\mu}(\omega) - I) = (1 + i\omega)^{-1} D(\omega), \quad \omega \in \mathbf{R},$$

so we end up with the given identity (2.1). ∎

The proof of Theorem 2 is a simple reduction to the uniformly continuous case:

Proof of Theorem 2. Let $x \in \mathbf{B}(\mathbf{R};\mathbf{C}^n)$ satisfy (1.1), and let $f \in \mathbf{AP}(\mathbf{R};\mathbf{C}^n)$. By Lemma 2.4, f satisfies (3.3). If m is a sufficiently large integer, then $e^{m*} * x \in \mathbf{BUC}(\mathbf{R};\mathbf{C}^n)$, and $e^{(m+1)*} * f \in \mathbf{BAP}(\mathbf{R};\mathbf{C}^n)$. Moreover, $y = e^{m*} * x$ satisfies

$$(4.2) \qquad y + a * y = g,$$

where $g = e^{(m+1)*} * f$. By Lemma 3.1, $x \in \mathbf{AP}(\mathbf{R};\mathbf{C}^n)$ if (and only if) $y \in \mathrm{BAP}(\mathbf{R};\mathbf{C}^n)$. In other words, it suffices to show that if $g \in \mathrm{BAP}(\mathbf{R};\mathbf{C}^n)$, then every solution $y \in \mathrm{BUC}(\mathbf{R};\mathbf{C}^n)$ of (4.2) belongs to $\mathrm{BAP}(\mathbf{R};\mathbf{C}^n)$. In the scalar case it follows e.g. from [11, Proposition 4.3] that this is indeed the case, because by (4.1), the Fourier transform of the kernel in (3.2) is invertible in all points $\omega \notin Z$, and Z was assumed to be countable. The matrix case is not considered in [11], but the proof given in [11] remains unchanged in the matrix case. This means that [11, Proposition 4.3] is valid also in the matrix case, and the proof is complete. ∎

In the proof of Theorem 4 we use the following fact, which is an immediate consequence of Theorem 1: The Fourier series of an almost periodic distribution solution x of (1.1) must be of the form

$$(4.3) \qquad \sum_{\substack{k=1 \\ \omega_k \notin Z}}^{\infty} D^{-1}(\omega_k) b_k e^{i\omega_k t} + \sum_{\lambda_k \in Z} a_k e^{i\lambda_k t},$$

where $\sum_{k=1}^{\infty} b_k e^{i\omega_k t}$ is the Fourier series of f, a_k belongs to the null space of $D(\lambda_k)$ if λ_k is not a characteristic exponent of f, and a_k is one of the infinitely many solutions to the equation $D(\lambda_k)a_k = b_j$, if $\lambda_k = \omega_j$ is a characteristic exponent of f (if this equation has no solution and Z is countable, then by Theorems 1 and 2, (1.1) has no bounded distribution solution).

<u>Proof of Theorem 4</u>. By Theorem 2, we know that x in Theorem 4 belongs to $\mathbf{AP}(\mathbf{R};\mathbf{C}^n)$. It follows from Theorem 1, (2.4), (4.3), and the fact that Z is finite that the Fouries series $\sum_{k=1}^{\infty} a_k e^{i\omega_k t}$ of x is absolutely convergent. The function $y(t)$ defined by $y(t) = \sum_{k=1}^{\infty} a_k e^{i\omega_k t}$, $t \in \mathbf{R}$, belongs to $\mathrm{BAP}(\mathbf{R};\mathbf{C}^n)$, and it has the

same Fourier series as x. The Fourier series of an almost periodic distribution determines it uniquely [10, p. 208], so in the distribution sense, $x = y$, and this is exactly what Theorem 4 claims. ∎

5. Proof of Theorem 3

Before proving Theorem 3, let us give a very short description of the space of Stepahoff almost periodic functions. Given p, $1 \le p < \infty$, we define the "Stephanoff" class of "bounded" function $S^p(R;C^n)$ to consist of those locally integrable functions f whose Stephanoff seminorm

$$|f|_{S^p} = \sup_{t \in R} \{ \int_t^{t+1} |f(s)|^p ds \}^{1/p} < \infty$$

is finite. We have $|f|_{S^p} \ge |f|_{S^1}$ for every $p > 1$ [2, p. 72], so in particular, $S^p(R;C^n) \subset S^1(R;C^n)$ for every $p > 1$. The Stephanoff almost periodic functions $SAP^p(R;C^n)$ of order p, $1 \le p < \infty$, can be defined in many different equivalent ways. The simplest definition is probably the one which says that $SAP^p(R;C^n)$ is the closure in $S^p(R;C^n)$ of $BAP(R;C^n)$. It is clear from this definition, and from the norm inequality given above, that $SAP^p(R;C^n) \subset SAP^1(R;C^n)$ for every $p > 1$.

In this work we shall really only need one elementary fact about $SAP^p(R;C^n)$, namely the following one:

<u>Lemma 5.1</u>. <u>Let</u> $1 \le p < \infty$, <u>let</u> $f \in SAP^p(R;C^n)$, <u>and define</u> e <u>as in</u> (3.1). <u>Then</u> $e * f \in BAP(R;C^n)$.

<u>Proof</u>. As $SAP^p(R;C^n) \subset SAP^1(R;C^n)$, it suffices to prove the lemma when $p = 1$. By the definition of $SAP^1(R;C^n)$, there is a sequence of functions $g_n \in BAP(R;C^n)$ converging to f in $S^1(R;C^n)$. We know from Lemma 3.2 that $e * g_n \in BAP(R;C^n)$ for all n. The straightforward computation

$$|e * f(t) - e * g_n(t)| = |\int_0^\infty e^{-t} [f(t-s) - g_n(t-s)] ds|$$

$$\leq \sum_{k=0}^\infty e^{-k} \int_k^{k+1} |f(t-s) - g_n(t-s)| ds$$

$$\leq \sum_{k=0}^\infty e^{-k} \|f - g_n\|_{S^1},$$

shows that $e * g_n$ converges to $e * f$ uniformly. This means that $e * f \in BAP(R;C^n)$ [3, Theorem V, p. 38]. ∎

Clearly, Lemmas 3.4 and 5.1 reduce Theorem 3 to a special case of the following theorem:

Theorem 6. *Let* $a \in L^1(R;C^{n\times n})$, *and suppose that the set where* $I - \hat{a}(\omega)$ *is not invertible is countable. Let* $f \in BAP(R;C^n)$. *Then every bounded distribution solution* x *of*

(5.1) $\quad x + a * x = f$

is Bohr almost periodic.

The problem can be even further simplified: We know already that if $x \in BUC(R;C^n)$ satisfies (5.1), and the other assumptions of Theorem 5 hold, then $x \in BAP(R;C^n)$ (cf. the proof of Theorem 2). Thus, it suffices to prove the following lemma:

Lemma 5.2. *Let* $a \in L^1(R;C^{n\times n})$, $f \in BUC(R;C^n)$, *and let* $x \in B(R;C^n)$ *satisfy* (5.1). *Then* $x \in BUC(R;C^n)$.

Proof. By the Riemann-Lebesgue lemma, $\hat{a}(\omega) \to 0$ as $\omega \to \pm\infty$, so we can find a number Ω such that $I - \hat{a}(\omega)$ is invertible for $|\omega| \geq \Omega$. By Wiener's Tauberian theorem, we can find a function $b \in L^1(R;C^{n\times n})$ satisfying

$$(5.2) \qquad [I + \hat{b}(\omega)][I + \hat{a}(\omega)] = I, \quad |\omega| \geq \Omega.$$

Let η be a scalar L^1-function whose Fourier transform $\hat{\eta}$ has compact support, and which satisfies $\hat{\eta}(\omega) \equiv 1$ for $|\omega| \leq \Omega + 1$, and let δ be the unit point mass at zero. Convolve (5.1) with $\delta - \eta$ to get

$$(5.3) \qquad (I\delta + a) * (x - \eta * x) = f - \eta * f.$$

The distribution Fourier transform of $x - \eta * x$ vanishes on $(\Omega-1,\Omega+1)$, so by (5.2),

$$(I\delta + b) * (I\delta + a) \; * (x - \eta * x) = x - \eta * x.$$

Therefore, if we convolve (5.3) with $I\delta + b$ we get

$$(5.4) \qquad x - \eta * x = f - \eta * f + b * f - b * \eta * f.$$

As $f \in BUC(R;C^n)$, the right hand side of (5.4) also belongs to $BUC(R;C^n)$, and so $x - \eta * x \in BUC(R;C^n)$. The Fourier transform of $\eta * x$ has compact support, and therefore $\eta * x \in BUC(R;C^n)$. This means that x itself belongs to $BUC(R;C^n)$, and the proof is complete. ∎

6. A Counterexample

In the same way as one defines $SAP^p(R;C^n)$ to be the closure of $BAP(R;C^n)$ in $S^p(R;C^n)$, one can define two more classes of almost periodic functions, i.e. the Weyl and Besicovitch classes, to be the closures of $BAP(R;C^n)$ in the "Weyl" and "Besicovitch" spaces. For each p, $1 \leq p < \infty$, we define $W^p(R;C^n)$ and $B^p(R;C^n)$ to be the set of locally integrable functions f whose Weyl seminorm

$$|f|_{W^p} = \lim_{\ell\to\infty} \sup_{t\in R} \left\{ \frac{1}{\ell} \int_t^{t+\ell} |f(s)|^p ds \right\}^{1/p},$$

respectively Besicovitch seminorm

$$|f|_{B^p} = \left\{ \limsup_{t\to\infty} \frac{1}{2t} \int_{-t}^{t} |f(s)|^p ds \right\}^{1/p}$$

is finite. Again, $|f|_{W^p} \geq |f|_{W^1}$ and $|f|_{B^p} \geq |f|_{B^1}$ for all $p > 1$ [2, p. 73], so $W^p(R;C^n) \subset W^1(R;C^n)$ and $B^p(R;C^n) \subset B^1(R;C^n)$. For each fixed p, we have $S^p(R;C^n) \subset W^p(R;C^n) \subset B^p(R;C^n)$. The Weyl almost periodic functions $WAP^p(R;C^n)$ and Besicovitch almost periodic functions $BAP^p(R;C^n)$ can be characterized by the fact that they are the closures of $BUC(R;C^n)$ in $W^p(R;C^n)$ and $B^p(R;C^n)$, respectively. Clearly, this means that $SAP^p(R;C^n) \subset WAP^p(R;C^n) \subset BAP^p(R;C^n)$, and that $WAP^p(R;C^n) \subset WAP^1(R;C^n)$ and $BAP^p(R;C^n) \subset BAP^1(R;C^n)$ for all $p > 1$.

With the new notations we can rewrite Theorem 5 into the following, slightly more general form:

Theorem 5′. Let $1 \leq p < \infty$, and let $f \in L^p(R;C^n)$ have compact support and satisfy $\int_R f(s)ds \neq 0$. *Then* $f \in WAP^p(R;C^n)$, *but its integral* $\int_{-\infty}^{t} f(s)ds$ *does not belong to* $BAP^p(R;C^n)$.

The first claim in Theorem 5′ is obvious, because trivially, if $f \in L^p(R;C^n)$, then $\|f\|_{W^p} = 0$, so $f \in WAP^p(R;C^n)$. The second claim is a consequence of the following fact:

Lemma 6.1. Let f *be continuous on* $\mathbf{R}$ *with walues in* C^n, *let the limits* $f(-\infty)$ *and* $f(\infty)$ *exist, and suppose that* $f(-\infty) \neq f(\infty)$. *Then* $f \notin BAP^p(R;C^n)$.

Proof. As $BAP^p(R;C^n) \subset BAP^1(R;C^n)$ for every $p > 1$, it suffices to consider the case $p = 1$.

Suppose that $f \in BAP^1(R;C^n)$. In this case Lemma 4 in [2, p.93] shows that f has a mean value

$$M(f) = \lim_{t\to\infty} \frac{1}{2t} \int_{-t}^{t} f(s)ds.$$

An obvious modification of Besicovitch's proof shows that not only does f have a mean value $M(f)$, it actually has both a left mean value

$$M^-(f) = \lim_{t\to\infty} \frac{1}{t} \int_{-t}^{0} f(s)ds$$

and a right mean value

$$M^+(f) = \lim_{t\to\infty} \frac{1}{t} \int_{0}^{t} f(s)ds.$$

Moreover, the two mean values are equal (use the fact that all functions in $BAP(R;C^n)$ have this property; see [3, p. 44]). Clearly, the left and right mean values of f in Lemma 6.1 exist, but they are not equal, and so $f \notin BAP^1(R;C^n)$. ∎

References

1. L. Amerio and G. Prouse, Almost-Periodic Functions and Functional Equations, Van Nostrand Reinhold, New York, 1971.

2. A. S. Besicovitch, Almost Periodic Functions, Cambridge University Press, Cambridge, 1932.

3. H. Bohr, Almost Periodic Functions, Chelsea, New York, 1947.

4. C. Corduneanu, Almost Periodic Functions, Interscience, New York, 1968.

5. R. Doss, On the almost periodic solutions of a class of integro-differential equations, Ann. of Math. (2) 81 (1965), 117 - 123.

6. A. M. Fink, Almost Periodic Differential Equations, Springer, Berlin, 1974.

7. Y. Katznelson, An Introduction to Harmonic Analysis, Dover, New York, 1976.

8. J. J. Levin and D. F. Shea, On the asymptotic behavior of the bounded solutions of some integral equations. I - III, J. Math. Anal. Appl. 37 (1972), 42 - 82; 288 - 326; 537 - 575.

9. L. H. Loomis, The spectral characterization of a class of almost periodic functions, Ann. of Math (2) 72 (1960), 362 - 368.

10. L. Schwartz, Théorie des distributions, Nouvelle éd., Hermann, Paris, 1966.

11. O. J. Staffans, On asymptotically almost periodic solutions of a convolution equation, Trans. Amer. Math. Soc. 266 (1981), 603 - 616.

OJS/jvs

SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

REPORT DOCUMENTATION PAGE

READ INSTRUCTIONS BEFORE COMPLETING FORM

1. REPORT NUMBER #2642	2. GOVT ACCESSION NO. AD-A139 312	3. RECIPIENT'S CATALOG NUMBER
4. TITLE (and Subtitle) On the Almost Periodicity of the Solutions of an Integrodifferential Equation		5. TYPE OF REPORT & PERIOD COVERED Summary Report - no specific reporting period
		6. PERFORMING ORG. REPORT NUMBER
7. AUTHOR(s) Olof J. Staffans		8. CONTRACT OR GRANT NUMBER(s) DAAG29-80-C-0041
9. PERFORMING ORGANIZATION NAME AND ADDRESS Mathematics Research Center, University of Wisconsin 610 Walnut Street Madison, Wisconsin 53706		10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS Work Unit Number 1 - Applied Analysis
11. CONTROLLING OFFICE NAME AND ADDRESS U. S. Army Research Office P.O. Box 12211 Research Triangle Park, North Carolina 27709		12. REPORT DATE February 1984
		13. NUMBER OF PAGES 20
14. MONITORING AGENCY NAME & ADDRESS(if different from Controlling Office)		15. SECURITY CLASS. (of this report) UNCLASSIFIED
		15a. DECLASSIFICATION/DOWNGRADING SCHEDULE

16. DISTRIBUTION STATEMENT (of this Report)

Approved for public release; distribution unlimited.

17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report)

18. SUPPLEMENTARY NOTES

19. KEY WORDS (Continue on reverse side if necessary and identify by block number)

Almost periodic, convolution equation, integrodifferential equation

20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

We discuss the almost periodicity of bounded solutions of the integrodifferential equation

$$x' + \mu * x = f .$$

Here x and f map $\mathbb{R}$ into $\mathbb{C}^n$, the prime denotes differentiation, μ is an n by n matrix valued finite measure on $\mathbb{R}$, and f is either an almost periodic distribution, or an almost periodic function in the sense of Bohr, Stephanoff, Weyl or Besicovitch. In the first three cases we giv. a simple

DD FORM 1 JAN 73 1473 EDITION OF 1 NOV 65 IS OBSOLETE

UNCLASSIFIED (continued)

SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

ABSTRACT (continued)

sufficient condition (countability of the set where the characteristic function of the kernel is not invertible) for bounded solutions to be almost periodic. This condition is no longer sufficient in the last two cases, as we show with a simple counterexample.